Lemma 2:

Let H be the Hamiltonian of a fermionic harmonic oscillator. Then we have

$$\operatorname{Tr} e^{-\beta H} = \int d\theta^* d\theta \langle -\theta | e^{-\beta H} | \theta \rangle e^{-\theta^* \theta}$$

Proof:

Inserting the completeness relation into the definition of a partition function, we get

$$Z(\beta) = \sum_{n=0,1} \langle n | e^{-\beta H} | n \rangle$$

$$= \sum_n \int d\theta^* d\theta \, e^{-\theta^* \theta} \langle n | \theta \rangle \langle \theta | e^{-\beta H} | n \rangle$$

$$= \sum_n \int d\theta^* d\theta (1 - \theta^* \theta) \big(\langle n | 0 \rangle + \langle n | 1 \rangle \theta \big) \cdot \big(\langle 0 | e^{-\beta H} | n \rangle + \theta^* \langle 1 | e^{-\beta H} | n \rangle \big)$$

$$= \sum_n \int d\theta^* d\theta (1 - \theta^* \theta) \Big[\langle 0 | e^{-\beta H} | n \rangle \langle n | 0 \rangle - \theta^* \theta \langle 1 | e^{-\beta H} | n \rangle \langle n | 1 \rangle + \theta \langle 0 | e^{-\beta H} | n \rangle \langle n | 1 \rangle + \theta^* \langle 1 | e^{-\beta H} | n \rangle \langle n | 0 \rangle \Big]$$

$$= \sum_n \int d\theta^* d\theta \, (1 - \theta^* \theta) \Big[\langle 0 | e^{-\beta H} | 0 \rangle - \theta^* \theta \langle 1 | e^{-\beta H} | 1 \rangle$$

$$+ \theta\langle 0|e^{-\beta H}|1\rangle - \theta^*\langle 1|e^{-\beta H}|0\rangle\Big]$$

$$= \int d\theta^* d\theta\, e^{-\theta^*\theta}\langle -\theta|e^{-\beta H}|\theta\rangle$$

$\square$

→ coordinate in the trace is over "anti-periodic" orbits, i.e. the Grassmann variable is θ at $t=0$ and $-\theta$ at $t=\beta$

→ have to impose an anti-periodic boundary condition over $[0,\beta]$ in trace.

<u>Path integral expression:</u>

Using the identity

$$e^{-\beta H} = \lim_{N\to\infty}(1-\beta H/N)^N$$

we find

$$Z(\beta) = \lim_{N\to\infty}\int d\theta^* d\theta\, e^{-\theta^*\theta}\langle -\theta|(1-\beta H/N)^N|\theta\rangle$$

$$= \lim_{N\to\infty}\int d\theta^* d\theta \prod_{k=1}^{N-1}\int d\theta_k^* d\theta_k\, e^{-\sum_{n=1}^{N-1}\theta_n^*\theta_n}$$

$$\times \langle -\theta|(1-\varepsilon H)|\theta_{N-1}\rangle\langle\theta_{N-1}|\cdots|\theta_1\rangle\langle\theta_1|(1-\varepsilon H)|\theta\rangle$$

$$= \lim_{N\to\infty} \int \prod_{k=1}^{N} d\theta_k^* d\theta_k \, e^{-\sum_{n=1}^{N} \theta_n^* \theta_n}$$

$$\times \langle \theta_N | (1 - \varepsilon H) | \theta_{N-1} \rangle \langle \theta_{N-1} | \cdots | \theta_1 \rangle \langle \theta_1 | (1 - \varepsilon H) | -\theta_N \rangle$$

where we put $\varepsilon = \beta/N$ and $\theta = -\theta_N = \theta_0$,
$\theta^* = -\theta_N^* - \theta_0^*$

$\longrightarrow$ each matrix element is evaluated as

$$\langle \theta_k | (1 - \varepsilon H) | \theta_{k-1} \rangle$$

$$= \langle \theta_k | \theta_{k-1} \rangle \left[1 - \varepsilon \frac{\langle \theta_k | H | \theta_{k-1} \rangle}{\langle \theta_k | \theta_{k-1} \rangle} \right]$$

$$\simeq \langle \theta_k | \theta_{k-1} \rangle e^{-\varepsilon \langle \theta_k | H | \theta_{k-1} \rangle / \langle \theta_k | \theta_{k-1} \rangle}$$

$$\langle \theta_k | \theta_{k-1} \rangle = e^{\theta_k^* \theta_{k-1}} \quad \text{last lecture}$$

$$\langle \theta_k | H | \theta_{k-1} \rangle = (H_{00} + \theta_k^* H_{10} + H_{01} \theta_{k-1} + \theta_k^* \theta_{k-1} H_{11}) e^{\theta_k^* \theta_{k-1}}$$

$$= e^{\theta_k^* \theta_{k-1}} e^{-\varepsilon \omega (\theta_k^* \theta_{k-1} - \frac{1}{2})}$$

$$= e^{\varepsilon \omega / 2} e^{(1 - \varepsilon \omega) \theta_k^* \theta_{k-1}}$$

$\longrightarrow Z(\beta)$

$$= \lim_{N\to\infty} e^{\beta \omega / 2} \prod_{k=1}^{N} \int d\theta_k^* d\theta_k \, e^{-\sum_{n=1}^{N} \theta_n^* \theta_n (1 - \varepsilon \omega) + \sum_{n=1}^{N} \theta_n^* \theta_{n-1}}$$

$$= e^{\beta\omega/2} \lim_{N\to\infty} \prod_{k=1}^{N} \int d\theta_k^* d\theta_k \, e^{-\sum_{n=1}^{N}\left[\theta_n^*(\theta_n-\theta_{n-1}) + \varepsilon\omega\theta_n^*\theta_{n-1}\right]}$$

$$= e^{\beta\omega/2} \lim_{N\to\infty} \prod_{k=1}^{N} \int d\theta_k^* d\theta_k \, e^{-\theta^\dagger B \theta}$$

where

$$\theta = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix}, \quad \theta^\dagger = (\theta_1^*, \theta_2^*, \ldots, \theta_N^*)$$

$$B_N = \begin{pmatrix} 1 & 0 & \cdots & 0 & -y \\ y & 1 & 0 & \cdots & 0 \\ 0 & y & 1 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & y & 1 \end{pmatrix}$$

with $y = -1 + \varepsilon\omega$

$$\rightarrow Z(\beta) = e^{\beta\omega/2} \lim_{N\to\infty} \det B_N$$

$$= e^{\beta\omega/2} \lim_{N\to\infty} \left[1 + (1 - \beta\omega/N)^N\right]$$

$$= e^{\beta\omega/2}(1 + e^{-\beta\omega})$$

$$= 2\cosh\tfrac{1}{2}\beta\omega \quad \checkmark$$

Grassmann path integral for Dirac fermions

In analogy with the generating functional for the scalar field

$$Z = \int \mathcal{D}\varphi \, e^{iS(\varphi)}$$

$$= \int \mathcal{D}\varphi \, e^{i\int d^4x \frac{1}{2}\left[(\partial\varphi)^2 - (m^2 - i\varepsilon)\varphi^2\right]}$$

we write for the Dirac field

$$Z = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \, e^{iS(\psi,\bar{\psi})}$$

$$= \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \, e^{i\int d^4x \, \bar{\psi}(i\partial\!\!\!/ - m + i\varepsilon)\psi}$$

Treating the integration variables ψ, $\bar{\psi}$ as Grassmann valued spinors, we obtain

$$Z = \int \mathcal{D}\psi \int \mathcal{D}\bar{\psi} \, e^{i\int d^4x \, \bar{\psi}(i\partial\!\!\!/ - m + i\varepsilon)\psi}$$

$$= C' \det(i\partial\!\!\!/ - m + i\varepsilon)$$

$$= C' e^{\operatorname{tr}\log(i\partial\!\!\!/ - m + i\varepsilon)}$$

where C' is some multiplicative constant.

$$\to \operatorname{tr}\log(i\not\partial - m)$$
$$= \operatorname{tr}\log \gamma^5(i\not\partial - m)\gamma^5$$
$$= \operatorname{tr}\log(-i\not\partial - m)$$
$$= \frac{1}{2}\left[\operatorname{tr}\log(i\not\partial - m) + \operatorname{tr}\log(-i\not\partial - m)\right]$$
$$= \frac{1}{2}\operatorname{tr}\log(\partial^2 + m^2)$$
$$\to Z = C' e^{\frac{1}{2}\operatorname{tr}\log(\partial^2 + m^2 - i\varepsilon)}$$

this gives the same vacuum energy as obtained before with the canonical formalism (even with same sign!)

Dirac propagator

Let us now introduce Grassmannian spinor sources η and $\bar{\eta}$:

$$Z[\eta,\bar{\eta}] = \int \mathcal{D}\psi\,\mathcal{D}\bar{\psi}\, e^{i\int d^4x\left[\bar{\psi}(i\not\partial - m)\psi + \bar{\eta}\psi + \bar{\psi}\eta\right]}$$

Completing the square as in the case of the scalar field, we get

$$\bar{\psi} K \psi + \bar{\eta}\psi + \bar{\psi}\eta$$
$$= (\bar{\psi} + \bar{\eta} K^{-1}) K (\psi + K^{-1}\eta) - \bar{\eta} K^{-1} \eta$$

$$\rightarrow \quad Z[\eta, \bar{\eta}] = C'' e^{-i\bar{\eta}(i\not{\partial}-m)^{-1}\eta}$$

In other words, the propagator $S(x)$ for the Dirac field is given by

$$(i\not{\partial} - m) S(x) = \delta^{(4)}(x)$$

$\rightarrow$ solution is:

$$i S_F(x) = \int \frac{d^4p}{(2\pi)^4} \frac{i e^{-ip\cdot x}}{\not{p} - m + i\varepsilon}$$

in agreement with the result before.

Feynman rules for fermions

Let us consider the theory of a scalar field interacting with a Dirac field:

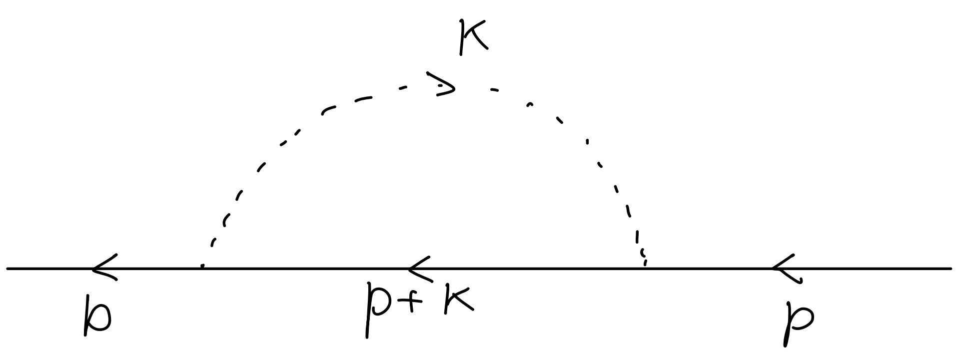
$$\mathcal{L} = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi + \frac{1}{2}\left[(\partial\varphi)^2 - \mu^2\varphi^2\right] - \lambda\varphi^4 + f\varphi\bar{\psi}\psi$$

→ generating functional:

$$Z[\eta,\bar\eta] = \int \mathcal{D}\psi\,\mathcal{D}\bar\psi\,\mathcal{D}\varphi\, e^{iS(\psi,\bar\psi,\varphi) + i\int d^4x\,(J\varphi+\bar\eta\psi+\bar\psi\eta)}$$

→ can be evaluated as a double series in the couplings λ and f using the following rules:

1) Draw a diagram with straight lines for the fermion and dotted lines for the boson, label each line with a momentum



2) Associate with each fermion line the propagator $\dfrac{i}{\not p - m + i\varepsilon} = i\,\dfrac{\not p + m}{p^2 - m^2 + i\varepsilon}$

3) Associate with each interaction vertex the coupling ig and the factor

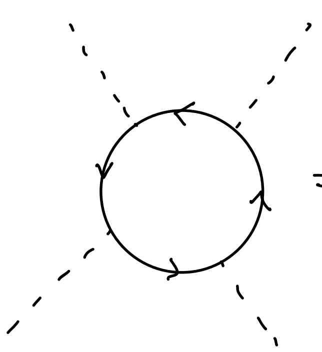
$$(2\pi)^4 \delta^{(4)}\left(\sum_{in} p - \sum_{out} p\right)$$ "momentum conservation"

4) Momenta associated with internal lines are to be integrated over with the measure

$$\int \left[\frac{d^4 p}{(2\pi)^4}\right]$$

5) External lines are to be amputated. For an incoming fermion line write $u(p,s)$ and for an outgoing fermion line $\bar{u}(p',s)$

6) A factor (-1) is to be associated with each closed fermion loop.



$$= \bar\psi \psi \bar\psi \psi \bar\psi \psi \bar\psi \psi$$

$$= (-1)\, tr\left[\psi\bar\psi \, \psi\bar\psi \, \psi\bar\psi \, \psi\bar\psi\right]$$

$$= (-1)\, tr\left[S_F S_F S_F S_F\right]$$